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A Teaching Proposal: Mechanical Analog of an Over-damped Josephson Junction

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The Josephson Junction (JJ)

In 1973 B. D. Josephson received the Nobel Prize for having predicted the d. c. and a. c. Josephson effects in a superconducting device consisting of two weakly coupled superconductors. This device was named "Josephson junction" (JJ). The dynamics of the superconducting phase difference ϕ across the junction is described by the Josephson equations [1]:

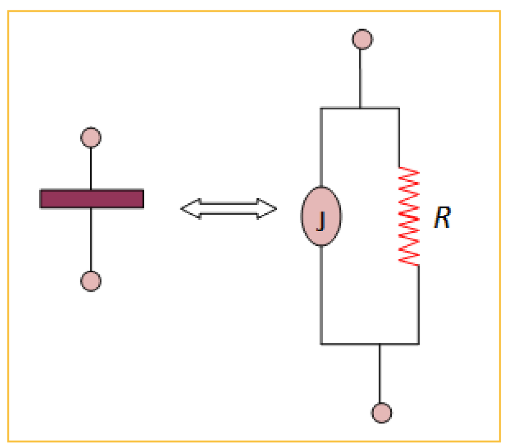

Fig. 1

$$I = I_J \sin\phi \text{, (1a)}$$

$$\frac{d\phi}{dt} = \frac{2e}{\hbar}V \text{, (1b)}$$

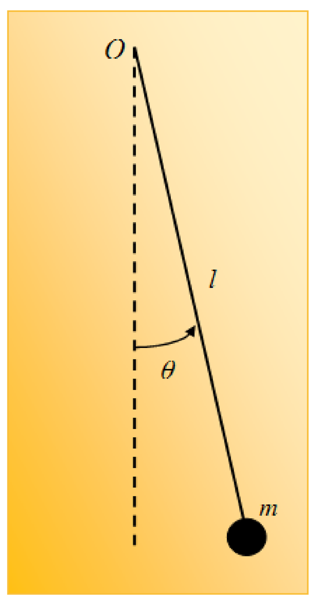

Fig. 2

where I is the current flowing through the junction (I_J being the maximum value that can flow in the zero-voltage state), $\hbar = h/2\pi$, h being Planck's constant, and V is the voltage across the two superconductors. In order to describe the dynamics of the superconducting phase difference ϕ in an over-damped JJ, a Resistively Shunted Junction model can be adopted [2]. In this model a purely superconducting element carrying a current I expressed in terms of ϕ as in Eq. (1a) is placed in parallel with a resistor of resistance R, as shown in fig. 1. By injecting a current I_B in the system and by invoking charge conservation, we may write:

$$\frac{V}{R} + I_J \sin\phi = I_B \text{, (2)}$$

where V is the voltage across the JJ. By expressing V in terms of ϕ as in Eq. (1b) and by introducing the dimensionless quantities $i_B = I_B/I_J$ and $\tau = \frac{2\pi R I_J}{\Phi_0}t$ we may rewrite Eq. (2) as follows:

$$\frac{d\phi}{d\tau} + \sin\phi = i_B \text{, (3)}$$

The above equation also represents the dynamics of an over-damped simple pendulum [3].

An Over-damped Pendulum

Let us consider the pendulum hinged in O and consisting of a massless rod of length l and a spherical body of radius R and mass m, as shown in fig. 2. This sphere is moving in a fluid of density ρ_F, so that it is subject to the buoyancy force. $m^* = m\left(1 - \frac{4\pi R^3}{3m}\rho_F\right)$ is the effective mass of the sphere, when buoyancy is taken into account. By setting $m_0(\tau) = \frac{M_0(\tau)}{m^* g(l+R)}$, where M0 is the applied torque and $\tau = \frac{m^* g}{6\pi\eta R(l+R)}t$ is a dimensionless time variable,

$$\text{for } \frac{m^* m g\left(l^2 + 2Rl + \frac{7R^2}{5}\right)}{(6\pi\eta R)^2 (l+R)^3} << 1 \text{, (4)}$$

we may write the following dynamical equation for the over-damped pendulum:

$$\frac{d\theta}{d\tau} + \sin\theta = m_0(\tau) \text{, (5)}$$

Constant Driving Moment

Let us take a constant forcing term of the over-damped pendulum: in this case we can obtain analytic solutions for the differential equation (5). for $m_0 < 1$, we obtain two constant solutions, one stable, one unstable, as it can be argued by means of the phase-plane analysis shown in fig. 3. The stable solution is given by:

$$\theta^* = \sin^{-1} m_0 \text{, (6)}$$

while the unstable solution is at $\theta = \pi - \theta^*$. The stability regime changes as the angle crosses the value $\theta = \pi/2$, as it can be noticed by analysing the sign of the derivative $d\theta/d\tau$ about these fixed points. For $m_0 = 1$ we have an half-stable solution: the pendulum may swing around O whenever an arbitrary small positive perturbation arises. For $m_0 > 1$, the function $\theta = \theta(\tau)$ is monotonically increasing, given that the curves in fig. 3 lie above the θ-axis and the derivative $d\theta/d\tau$ is always positive. In this "running state" we solve the ordinary differential equation (5) by the method of separation of variables [2], by writing:

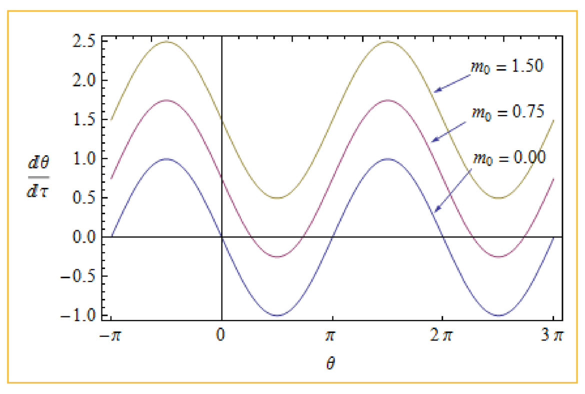

Fig. 3 Phase-plane analysis for the over-damped pendulum. The constant forcing term is $m0 = 0.0$ (bottom curve), $m0 = 0.75$ (middle curve), and $m0 = 1.50$ (top curve).

$$\int_{\theta_0}^{\theta(\tau)} \frac{d\theta}{m_0 - \sin\theta} = \tau \text{, (7)}$$

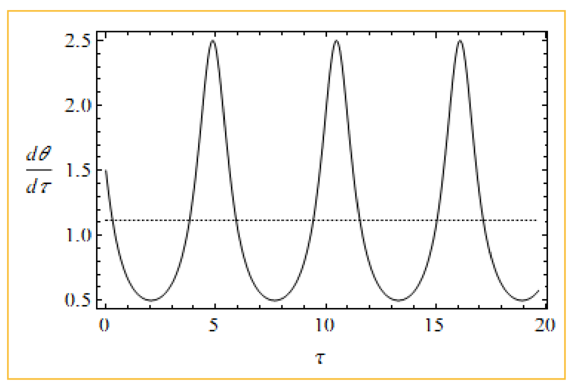

Fig. 4 Normalized time dependence of the angular frequency (full line) of an over-damped pendulum subject to a constant forcing equal to $m_0 = 1.50$.

where $\theta = \theta(0)$.

By finding the function $\theta = \theta(\tau)$ we may calculate the time average $\left\langle\frac{d\theta}{d\tau}\right\rangle$ of the angular frequency $\frac{d\theta}{d\tau}$ as a function of the constant forcing term m_0. This analysis is important, given that the m0 versus $\left\langle\frac{d\theta}{d\tau}\right\rangle$ curves correspond to the normalized current iB versus average voltage $\left\langle\frac{d\phi}{d\tau}\right\rangle$ characteristics of an over-damped Josephson junction. We notice that the function is $\frac{d\theta}{d\tau}$ periodic with period equal to $T = \frac{2\pi}{\sqrt{m_0^2 - 1}}$.

On the other hand, the time-averaged value of $\frac{d\theta}{d\tau}$ can be calculated as follows:

$$\left\langle\frac{d\theta}{d\tau}\right\rangle = \frac{1}{T}\int_0^T \frac{d\theta}{d\tau}d\tau = \frac{\theta(T) - \theta(0)}{T} = \frac{2\pi}{T} \text{, (8)}$$

so that it is proven that the average value of the angular frequency curves is $\sqrt{m_0^2 - 1}$. From Eq. (8) we can then argue that:

$$m_0 = \sqrt{1 + \left\langle\frac{d\theta}{d\tau}\right\rangle^2} \text{, (9)}$$

for $m_0 < 1$ the pendulum is in static equilibrium, so that $\left\langle\frac{d\theta}{d\tau}\right\rangle = 0$. The same happens in a Josephson junction: when the value of the normalized bias current i_B is less than one, the junction is said to be in the superconducting or zero-voltage state. Therefore, no current flows in the resistive branch of the RSJ model in fig. 1, so that the curve climbs vertically from 0 to 1 just as shown in fig. 5. However, when $i_B > 1$, the resistive branch is activated and a finite voltage appears across the junction, in the way described in fig. 5. We also notice that the m_0 versus $\left\langle\frac{d\theta}{d\tau}\right\rangle$ curve presents the oblique asymptote $m_0 = \left\langle\frac{d\theta}{d\tau}\right\rangle$. In fact, for large enough values of m_0, this driving moment becomes predominant with respect to the nonlinear sine term in Eq. (5), thus justifying the observed asymptotic.

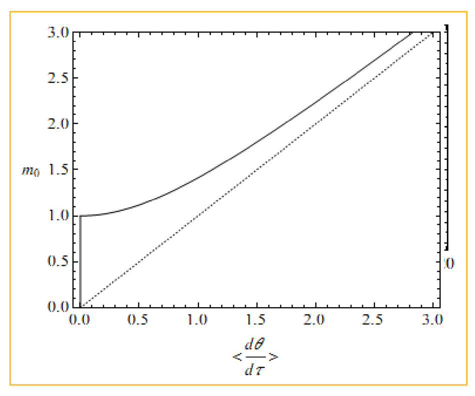

Fig. 5 Normalized forcing term versus the time average of the angular frequency (full line) of an over-damped pendulum.

Conclusions

The present work is devoted to teachers who are willing to actually construct the mechanical analog. The properties of an over-damped Josephson junction have been analysed by means of a mechanical analogue: an over-damped pendulum: being the physical properties of a pendulum more familiar to students, the Josephson junction dynamics in the over-damped limit may be derived by analogy.

References

[1] B. D. Josephson, Phys. Lett. 1, 251 (1963).

[2] A. Barone and G. Paternò, Physics and applications of the Josephson Effect (New York, Wiley, 1982).

[3] D. B. Sullivan and J. E. Zimmerman, Am. J. Phys. 39, 1504 (1971).