Lezione 3 del 31/10/2023

Equazione logaritmica

L'incognita compare nell'argomento di uno o più logaritmi

$$\boxed{\log_a x = b} \qquad a>0,\ a\neq 1,\ b\in\mathbb{R},\ x>0$$

base argomento

Il logaritmo è l'esponente da dare alla base per ottenere l'argomento

$$\boxed{a^b = x}$$

$$\log_3 x = 1 \iff x = 3^1 \qquad \boxed{x > 0}$$

$$\log_a f(x) = \log_a g(x) \iff \begin{cases} f(x) = g(x) \\ f(x) > 0 \\ g(x) > 0 \end{cases} \quad C.E.$$

$$\log_a f(x) = \log_b g(x) \iff \begin{cases} f(x) > 0 \\ g(x) > 0 \end{cases} \quad C.E.$$

$$f(x) = a^{\log_b g(x)}$$

- $\log_4 (x+2) = 1 \iff \log_4 (x+2) = \log_4 4$

$$\begin{cases} x+2 = 4 \iff x = 2 \\ x+2 > 0 \quad \text{C.E.} \quad \boxed{x > -2} \end{cases}$$

- $\log_3 (x^2-13) = 1$

C.E. $\quad x^2 - 13 > 0 \qquad x = \pm\sqrt{13}$

$x < -\sqrt{13} \cup x > \sqrt{13}$

$x^2 - 13 = 3 \qquad x^2 = 16$

$\boxed{x = \pm 4}$

$\log_a a = 1$

$\log_a 1 = 0$

$\log_a x^n = n \log_a x$

$\log_a^n x \neq \log_a x^n$

$\log_a x + \log_a y = \log_a x \cdot y$

$\log_a x - \log_a y = \log_a \frac{x}{y}$

- $\log_5(-x+2) = \log_{\frac{1}{5}} 3$

$\iff \log_5(-x+2) = -\log_5 3$

$\iff \log_5(-x+2) = \log_5 1/3$

$\iff -x+2 = \frac{1}{3} \iff \begin{cases} x = \frac{5}{3} \\ -x+2>0 \end{cases}$? ok

$x < 2$

$3^{-1} = \frac{1}{3}$

- $\log^2 x + \log x - 2 = 0$

Pongo $\boxed{\log x = t}$

$t^2 + t - 2 = 0$

$\Delta = 1+8 = 9$ $\qquad t = \frac{-1 \pm 3}{2} = \begin{matrix} -2 \\ 1 \end{matrix}$

$\log x$ base e

$\mathrm{Log}\, x$ base 10

$\ln x$

$\log x = -2 \quad , \quad \log x = 1$

$\Updownarrow \qquad\qquad\qquad \Updownarrow$

$x = e^{-2} \quad , \quad x = e$ $\qquad \underline{x > 0 \quad C.E}$

ok!

Disequazione

Tipo $\log_a f(x) \gtrless \log_a g(x)$

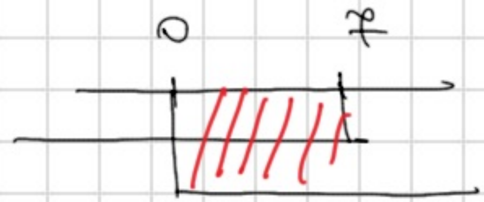
$a > 1$: $\log_a m < \log_a n \iff m < n$

$0 < a < 1$: $\log_a m < \log_a n \iff m > n$

La relazione d'ordine tra gli argomenti si inverte.

$\log_3 7 > \log_3 x$ $\qquad a = 3$

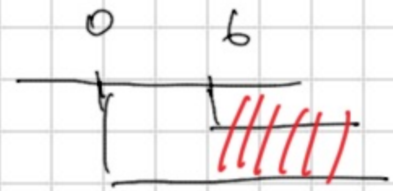
$$\begin{cases} x < 7 \\ x > 0 \end{cases}$$



$\boxed{0 < x < 7}$

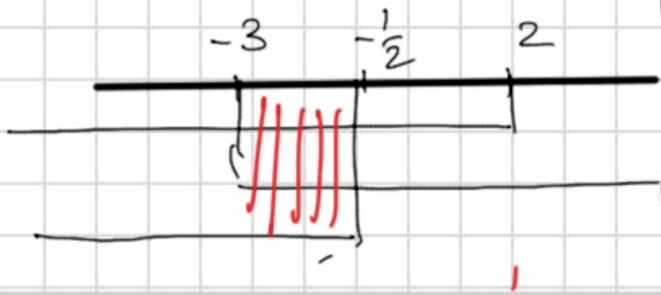
- $\log_{\frac{1}{4}} x < \log_{\frac{1}{4}} 6$

$$\begin{cases} x > 6 \\ x > 0 \end{cases} \iff x > 6$$



- $\log_{\frac{1}{3}} (2-x) < \log_{\frac{1}{3}} (x+3)$

$$\begin{cases} 2-x > 0 \\ x+3 > 0 \\ 2-x > x+3 \end{cases} \iff \begin{cases} x < 2 \\ x > -3 \\ -2x > 1 \end{cases} \quad x < -\frac{1}{2}$$



$-3 < x < -\frac{1}{2}$

$$\log_3(x-1) - \log_{\frac{1}{3}}(2-x) < -1$$

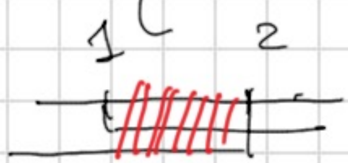
$$\log_3(x-1) + \log_3(2-x) < -1$$

$$\log_3 (x-1)(2-x) < -1$$

$$\begin{cases} (x-1)(2-x) < \frac{1}{3} \\ x-1>0 \\ 2-x>0 \end{cases} \iff \begin{cases} 2x - x^2 - 2 + x < \frac{1}{3} \\ = \\ = \end{cases}$$

$$\Delta < 0$$

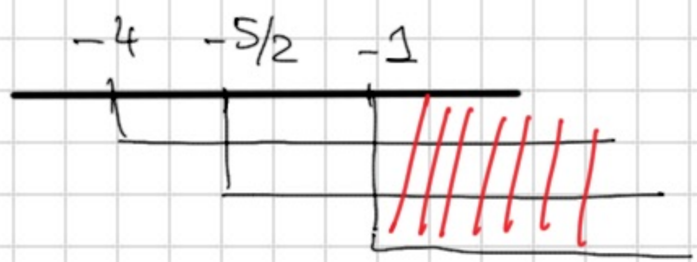
$$\begin{cases} -x^2 + 3x - 2 - \frac{1}{3} < 0 \\ = \\ = \end{cases} \iff \begin{cases} \boxed{3x^2 - 9x + 7 > 0} \\ x > 1 \\ x < 2 \end{cases}$$



$$\boxed{1 < x < 2}$$

- $\log_{\frac{1}{2}}(2x+5) < \log_{\frac{1}{2}}(x+1)$

$$\begin{cases} 2x+5 > x+1 \\ 2x+5 > 0 \\ x+1 > 0 \end{cases} \Leftrightarrow \begin{cases} x > -4 \\ x > -5/2 \\ x > -1 \end{cases}$$



$$\boxed{x > -1}$$

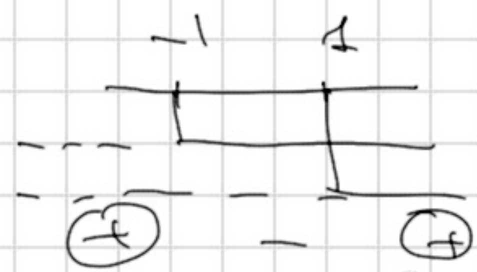
$$\log_{\frac{2}{5}}\left(\frac{x+1}{x-1}\right) \geq 0$$

$$\begin{cases} * \ \dfrac{x+1}{x-1} \leq 1 \to \left(\dfrac{2}{5}\right)^{0} \\[2ex] \dfrac{x+1}{x-1} > 0 \quad ** \end{cases}$$

$$* \ \frac{x+1}{x-1} - 1 \leq 0$$

$$\frac{\cancel{x}+1 - \cancel{x}+1}{x-1} \leq 0 \iff \frac{2}{x-1} \leq 0 \iff x < 1$$

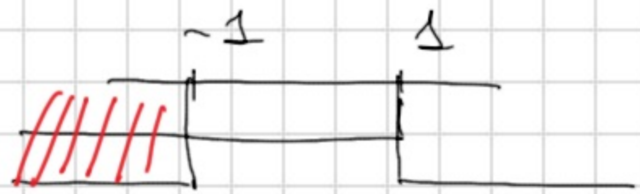
$$** \ \left[\begin{array}{l} x+1 > 0 \\ x-1 > 0 \end{array}\right. \iff \left[\begin{array}{l} x > -1 \\ x > 1 \end{array}\right.$$



$$x < -1 \cup x > 1$$

$$\begin{cases} x < 1 \\ x < -1 \cup x > 1 \end{cases}$$

$$\begin{cases} x<1 \\ x<-1 \cup x>1 \end{cases}$$



$$\boxed{x<-1}$$

- $\log_3^2 x + \log_3 x - 6 > 0$

Pongo $\log_3 x = t$

$t^2 + t - 6 > 0$

$\Delta = 1 + 24 = 25$

$t = \frac{-1 \pm 5}{2}$ → $t_1 = -3$, $t_2 = 2$

$t < -3 \cup t > 2$

⇓

$\log_3 x < -3 \cup \log_3 x > 2$

$$\begin{cases} x<\frac{1}{27} \cup x>9 \\ x>0 \end{cases}$$

1.8.5 Esercizi proposti

Risolvere le seguenti disequazioni logaritmiche:

1. $\log_3(2x+1) > 0$ $\mathbb{R}^+$
2. $\log_5(x-2) < 0$ $]2, 3[$
3. $\log_{\frac{1}{2}}(3x+5) < 0$ $]-\frac{4}{3}, +\infty[$
4. $\log_{\frac{1}{2}}(3x+5) < 1$ $]-\frac{3}{2}, +\infty[$
5. $\text{Log }(x-3) < 1$ $]3, 13[$
6. $\text{Log }(x^2-15x) > 2$ $]-\infty, -5[\cup]20, -\infty[$
7. $\log_3(\sqrt{x-1}-2) < 0$ $]5, 10[$
8. $\log_3\log_3(2x-5) < 0$ $]3, 4[$
9. $\text{LogLog}(x^2-6) < 0$ $]-4, -\sqrt{7}[\cup]\sqrt{7}, 4[$
10. $\text{Log}^2 x - 4\text{Log }x > 0$ $]0, 1[\cup]10^4, +\infty[$

11. $\log_3(2-3^x)+x>0$ $\quad\emptyset$

12. $(2\log_2 x-1)\log_{\frac{1}{2}} 5 \le 0$ $\quad[\sqrt{2},+\infty[$

13. $\log_3 x^2 - 2\log_9 x - 2 < 0$ $\quad]0,9[$

14. $\log_5 \frac{1}{x} - \log_{25} x^2 < 2$ $\quad]\frac{1}{5},+\infty[$

15. $2\log_2 x^3 - \log_4 x^2 + 1 < 0$ $\quad\left]0,\sqrt[5]{\frac{1}{2}}\right[$

16. $\log_3 \log_{\frac{1}{3}}(1+3x) > 0$ $\quad]-\frac{1}{3},-\frac{2}{9}[$

17. $\log_2^2 x - \log_2 x > 0$ $\quad]0,1[\,\cup\,]2,+\infty[$

18. $\log_{\frac{1}{3}}^2 x + \log_{\frac{1}{3}} x - 2 \le 0$ $\quad[\frac{1}{3},9]$

19. $\log_5^2 x + \log_5 x - 2 > 0$ $\quad]0,\frac{1}{25}[\,\cup\,]5,+\infty[$

20. $\log_3^2 x + 2\log_3 x - 3 < 0$ $\quad]\frac{1}{27},3[$

Equazioni esponenziali

L'incognita compare soltanto nell'esponente di una o più potenze.

- $a^x = k$
- $a^{f(x)} = a^{g(x)}$
- $a^{f(x)} = b^{g(x)}$
- $f(a^x) = k$

esponente

$$a^x = k$$

base

$$\boxed{0 < a < 1}$$

$$\boxed{a > 1}$$

$$a^x = k \qquad a > 0, \quad k \in \mathbb{R}, \quad x \in \mathbb{R}$$

- IMPOSSIBILE se $k \leq 0$, oppure se $k \neq 1$ e $a = 1$
- INDETERMINATA se $a = 1$, $k = 1$
- DETERMINATA se $a > 0$, $a \neq 1$, $k > 0$

- $9^{x+2} = 3 \Leftrightarrow 3^{2(x+2)} = 3^{1}$

  $\Leftrightarrow 2x+4 = 1 \Leftrightarrow x = -\frac{3}{2}$

- $4^{5x+2} = 16^{2x+1} \Leftrightarrow 4^{5x+2} = 4^{2(2x+1)}$

  $\Leftrightarrow 5x+2 = 4x+2 \Leftrightarrow x = 0$

- $\left(\frac{1}{36}\right)^{-3x+2} = 6^{2x+3} \Leftrightarrow 6^{-2(-3x+2)} = 6^{2x+3}$

  $\Leftrightarrow 6x-4 = 2x+3 \Leftrightarrow 4x = 7 \Leftrightarrow x = \frac{7}{4}$

- $4^x = 5 \iff \log 4^x = \log 5$

$\iff x \cdot \log 4 = \log 5 \iff \boxed{x = \dfrac{\log 5}{\log 4}}$

$\Rightarrow x = \boxed{\log_4 5}$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

- $3^{x^3 - x^5} + 81 = 0$

$\iff 3^{x^3 - x^5} = -81$ impossible

$$2^{-x+3} = 9$$

$$\Rightarrow \log 2^{-x+3} = \log 9 \iff (-x+3)\log 2 = \log 9$$

$$\iff -x\log 2 + 3\log 2 = \log 9$$

$$\iff -x = \frac{\log 9 - 3\log 2}{\log 2} \iff x = \frac{3\log 2 - \log 9}{\log 2}$$

$$\Updownarrow$$

$$-x+3 = \log_2 9 \iff x = -\log_2 9 + 3$$

Disequazioni esponenziali:

$$a^{f(x)} \gtrless a^{g(x)}$$

$a > 1$: $\quad a^m < a^n \iff m < n$

$0 < a < 1$: $\quad a^m < a^n \iff m > n$

$$25^x > 5 \iff 5^{2x} > 5 \iff 2x > 1$$

$$\iff x > \frac{1}{2}$$

$$\left(\frac{1}{2}\right)^x > \left(\frac{1}{2}\right)^{-2} \iff x < -2$$

$$2^{x+1} \geq 5^{1-x}$$

$$\log 2^{(x+1)} \geq \log 5^{(1-x)}$$

$$(x+1)\log 2 \geq (1-x)\log 5$$

$$x\log 2 + \log 2 \geq \log 5 - x\log 5$$

$$x\log 2 + x\log 5 \geq \log 5 - \log 2$$

$$x(\log 2 + \log 5) \geq \log 5 - \log 2$$

$$x \geq \frac{\log 5 - \log 2}{\log 2 + \log 5}$$

$$a^{x+y} = a^x \cdot a^y$$

$$a^{x-y} = \frac{a^x}{a^y}$$

$$(a^x)^y = a^{xy}$$

$$a^1 = a$$

$$a^0 = 1$$

$$3^{x+1} - 3^{x-2} + 3^x \geq 35$$

$$3 \cdot 3^x - \frac{3^x}{3^2} + 3^x \geq 35$$

$$27 \cdot 3^x - 3^x + 9 \cdot 3^x \geq 35$$

$$\cancel{9 \cdot 3^x = 27^x}$$

$$\cancel{35} \cdot 3^x \geq \cancel{35} \cdot 3^2 \iff 3^x \geq 3^2 \iff x \geq 2$$

- $3^{2x+1} - 3^{2+x} + 6 \leq 0$

$3 \cdot 3^{2x} - 9 \cdot 3^{x} + 6 \leq 0$

dis di 2° grado

$3^x = t$

$3t^2 - 9t + 6 \leq 0$

$\Delta = 81 - 72 = 9$

$t = \frac{9 \pm 3}{6}$ $t_1 = 1$, $t_2 = 2$

$\Rightarrow 1 \leq t \leq 2$

$\Downarrow$

$1 \leq 3^x \leq 2$

$\Downarrow$

$0 \leq x \leq \log_3 2$

$3^x = 1 \Leftrightarrow x = 0$

$3^x = 2 \Leftrightarrow x = \log_3 2$

$$2 \cdot 3^x - 9^x > 1$$

$$2 \cdot 3^x - 3^{2x} - 1 > 0$$

$$-3^{2x} + 2 \cdot 3^x - 1 > 0$$

$$3^{2x} - 2 \cdot 3^x + 1 < 0$$

Pongo $3^x = t$

$$t^2 - 2t + 1 < 0$$

$$(t-1)^2 < 0$$

La diseq. non ha soluzione

2. $\dfrac{3^{x+1}}{27^{2x}} < \dfrac{1}{3^{x^2+5}}$ $\quad]2,3[$

3. $4\sqrt{2} < \dfrac{1}{\sqrt{8^x}}$ $\quad]-\infty,-\frac{5}{3}[$

4. $\sqrt{2^x} \geq 8 \cdot \sqrt[3]{4^{x-1}}$ $\quad]-\infty,-14]$

5. $2^{\frac{2x+4}{x}} < \left(\dfrac{1}{4}\right)^{-2}$ $\quad \mathbb{R}^- \cup]2,+\infty[$

6. $\dfrac{3^{x-1}}{27^{1-x}} < \dfrac{9}{3^{2+x}}$ $\quad]-\infty,\frac{4}{5}[$

7. $\dfrac{2^{x-1} \cdot 4^{1+x}}{6^{1-x}} < 3$ $\quad \left]-\infty,\frac{\operatorname{Log} 9}{\operatorname{Log} 48}\right[$

8. $3^x + 3^{x+1} + 3^{x+2} \geq 39$ $\quad [1,+\infty[$

9. $2^{2x+1} + 4^{x-1} + 4^x < 13$ $\quad]-\infty,1[$

10. $3^{x+1} - 3^{x-2} + 3^x \geq 35$ $\quad [2,+\infty[$

11. $4^x - 3 \cdot 2^x + 2 < 0$ $\quad]0,1[$

12. $5^x - 4 \geq 5^{1-x}$ $\quad [1,+\infty[$

13. $3 \cdot 2^x - 2^{-x} - 2 > 0$ $\quad \mathbb{R}^+$

14. $2^x + \dfrac{4}{2^x} \geq 4$ $\quad \mathbb{R}$

15. $\dfrac{2^x}{2^x+1} + \dfrac{2^x}{2^x+4} \leq 1$ $\quad]-\infty,1]$

16. $\dfrac{4 \cdot 7^{x-1}}{21+\sqrt{7^x}} \geq 1$ $\quad [2,+\infty[$

• $3^{4x} - 3^{3x} - 7\cdot 3^{2x} + 3^{x} + 6 < 0$

Pongo $3^{x} = t$

$t^4 - t^3 - 7t^2 + t + 6 < 0$ $\qquad \pm 1 ; \pm 2 ; \pm 3$

$P(1) = \cancel{1} - \cancel{1} - \cancel{7} + \cancel{1} + \cancel{6} = 0$

$$\begin{array}{c|cccc|c} & t^4 & & & & \\ & 1 & -1 & -7 & 1 & 6 \\ 1 & & 1 & 0 & -7 & -6 \\ \hline & 1 & 0 & -7 & -6 & \cancel{0} \\ & t^3 & t^2 & t & & \end{array}$$

$(t-1)(t^3 - 7t - 6) < 0$

$$t^3 - 7t - 6 = 0 \qquad \pm 1$$

$$P(1) = 1 - 7 - 6 \neq 0$$

$$P(-1) = -1 + 7 - 6 = 0$$

$$\begin{array}{c|ccc|c} & 1 & 0 & -7 & -6 \\ -1 & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & // \end{array}$$

$$(t+1)(t^2 - t - 6)$$

$$(t-1)(t+1)(t^2 - t - 6) < 0$$

$$\begin{cases} t-1 > 0 \\ t+1 > 0 \\ t^2 - t - 6 > 0 \end{cases}$$

To be continued